Non-relativistic particle in an external magnetic field

Maxwell's equations (in cgs)

1. \( \nabla \cdot \mathbf{B} = 0 \)  \( \) (Homogeneous eqns)  \( \nabla \times \mathbf{B} = \frac{4\pi}{c} (\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}) \)  \( \) (inhomogeneous)

2. \( \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \)  \( \) "EM Potentials"  \( \nabla \cdot \mathbf{E} = 4\pi \mathbf{S} \)  \( \) "dynamics"

Lorentz force equation \( \mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}) \)  \( q = \text{charge of particle}. \)

Helmholtz theorem: for any \( C^2 \) function \( \mathbf{f} \) with \( \nabla \cdot \mathbf{f} = 0 \), we can write \( \mathbf{f} = -\nabla \phi + \nabla \times \mathbf{A} \)  \( \) (generalizes to Hadamard decomposition)

**EM Potentials**

1. \( \nabla \cdot \mathbf{B} = 0 \)  \( \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \)

2. \( \nabla \times (\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}) = 0 \)  \( \Rightarrow \mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \)

\( \phi \) & \( \mathbf{A} \) are not unique. Remembering that \( \nabla \cdot \nabla \phi = 0 \)

we see that \( \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \phi \) then \( \mathbf{B}' = \mathbf{B} \)

Now let \( \phi \rightarrow \phi + \alpha \), then \( \mathbf{E} \rightarrow \mathbf{E}' = \mathbf{E} - \nabla \alpha - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \nabla \phi \)

So if \( \alpha = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \) then \( \mathbf{E} \) is also invariant.

**Gauge transforms:**

\[ (\phi, \mathbf{A}) \rightarrow (\phi, \mathbf{A}) + (-\frac{1}{c} \frac{\partial}{\partial t}, \nabla) \phi \]

(note the natural 4-vector behaviour)
Classical action:

\[ L = \frac{m}{2} \dot{\vec{x}}^2 - q \phi + \frac{q}{c} \dot{\vec{v}} \cdot \vec{A} = L(r, \dot{r}, t) \]

\[ \dot{A} = \vec{A}(r, t) \quad \& \quad \phi = \phi(r, t) \]

\& that the potential energy term is velocity dependent.

Check gauge invariance:

\[ L \rightarrow L' = \frac{m}{2} \dot{\vec{x}}^2 - q \left( \phi - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{A} \right) + \frac{q}{c} \dot{\vec{v}} \cdot (\vec{A} + \nabla \phi) \]

\[ = L + \frac{q}{c} \left( \frac{\partial}{\partial t} \vec{v} + \frac{\partial}{\partial t} \frac{\partial}{\partial r} \right) = L + \frac{q}{c} \frac{\partial}{\partial t} \vec{v} \]

\[ \Rightarrow S \rightarrow S + \int_{0}^{\infty} \frac{q}{c} \frac{\partial}{\partial t} \vec{v} \]

Equations of motion:

Generalized momentum:

\[ \vec{p} = \frac{d}{dt} \vec{r} = m \vec{v} + \frac{q}{c} \vec{A} \quad \text{(Leads to minimal coupling procedure in QM & QFT)} \]

**Force:**

\[ \vec{F} = \frac{d}{dt} \vec{p} = -q \nabla \phi + \frac{q}{c} \nabla (\nabla \cdot \vec{A}) \]

\[ = q \vec{E} + \frac{q}{c} (\partial_t \vec{A} + \nabla (\nabla \cdot \vec{A})) \]

\[ 0 = \frac{d}{dt} \vec{p} - \vec{F} = 0 \]

\[ \Rightarrow 0 = m \ddot{\vec{r}} + \frac{q}{c} \frac{d}{dt} \vec{A} - q \vec{E} - \frac{q}{c} (\partial_t \vec{A} + \nabla (\nabla \cdot \vec{A})) \]

\[ = m \ddot{\vec{r}} - q \vec{E} + \frac{q}{c} (\nabla \cdot \nabla \cdot \vec{A}) \]

now \( \nabla \cdot \vec{A}_i - \nabla \cdot \vec{A}_j = \nabla \times (\nabla \times \vec{A}) \)

\[ \Rightarrow \nabla \times \vec{A} = \frac{q}{c} \vec{E} + \nabla \times \vec{B} \]

\[ E = \vec{p} \cdot \dot{\vec{r}} - L = \dot{r} \frac{d}{dt} \frac{d}{dr} - L \quad : \text{Any terms linear in velocity cannot contribute.} \]

\[ E = \frac{1}{2} m \dot{r}^2 + q \phi \]
Constant Electric Field

\[ \vec{E} = \text{const}, \quad \vec{B} = 0 \]

Now \( \vec{B} = 0 \Rightarrow \nabla \times \vec{A} = 0 \Rightarrow \vec{A} = \nabla \phi, \quad \text{ie is pure gauge} \)

So we can choose \( \vec{A} = 0, \quad \vec{E} = -\nabla \phi(\vec{r}) \) \quad \( (\nabla \cdot \vec{E} = 0 \Rightarrow \nabla \cdot \vec{\phi} = 0) \)

Thus \( \vec{E} = \text{const} \Rightarrow \nabla \phi = \text{const} \Rightarrow \phi = \vec{n} \cdot \vec{x} + \vec{k}, \quad \vec{n}, \vec{k} \text{ are const.} \) \quad \( (** \text{see below}**) \)

Equation of motion are \( m \ddot{\vec{r}} = q \vec{E} = \text{const.} \) \quad \( (\text{ie constant force problem}) \)

\[ \ddot{\vec{r}} = \vec{v}_0 + \vec{v}_0 (t-t_0) + \frac{1}{2m} (t-t_0)^2 \vec{E} \]

Compared with Assignment 1, Question 1a)

Also note that \( \phi = -\vec{E} \cdot \vec{r} + \text{const} \)

Check: \( -\nabla \phi = +\nabla (\vec{E} \cdot \vec{r}) + 0 \)

\[ \Rightarrow -\partial_i \phi = E_j \partial_j \vec{r} = E_j \delta_{ij} = E_i \]

\( (** \text{So above we had } \vec{n} = -\vec{E} **) \)
Constant Magnetic field $\vec{E} = 0$, $\vec{B} = \text{const.}$

We can choose $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$, $\phi = 0$

Check:

$\nabla \times \vec{A} = \frac{1}{2} \nabla \times \left( \vec{B} \times \vec{r} \right)$

$= \frac{1}{2} \left( \vec{r} \partial_x \vec{B} + \vec{B} \left( \vec{r} \cdot \vec{\nabla} \right) - \vec{B} \left( \vec{\nabla} \cdot \vec{r} \right) - \vec{r} \left( \vec{\nabla} \cdot \vec{B} \right) \right)$

$= \frac{1}{2} \left( \vec{B} \partial_x + \partial_x \vec{B} \right)$

$= \vec{B}$

$\nabla \cdot \vec{A} = \frac{1}{2} \partial_i \varepsilon_{ijk} \vec{B}_j \hat{r}_k = \frac{1}{2} \vec{B} \varepsilon_{ijk} \delta_{jk} = 0$

$m \ddot{\vec{r}} = 2 \frac{q}{mc} \vec{v} \times \vec{B}$, a homogeneous, 1st order DE in $\vec{v} = \dot{\vec{r}}$

Note that $\frac{1}{2} \frac{d}{dt} \vec{v}^2 = \frac{1}{2} \frac{q}{mc} \vec{v} \cdot \left( \vec{v} \times \vec{B} \right) = 0$

$\Rightarrow \vec{v}^2 = \text{const} = V_x^2 + V_y^2 \Rightarrow \text{Conservation of energy}$

New choose (rotate) axis s.t. $\vec{B} = (0, 0, B)$

$\nabla \times \vec{B} = (0, 0, B)$

$\Rightarrow \nabla \times \vec{B} = (V_x \hat{y} - V_y \hat{x}, 0)$

Equation $\Rightarrow \dot{\vec{v}} = \frac{qB}{mc} \vec{v} \times \vec{B}$ $\Leftrightarrow (\dot{V}_x, \dot{V}_y, \dot{V}_z) = \omega (V_y, -V_x, 0)$

$\omega = \frac{qB}{mc}$

Choose $V_z(0) = 0$, $\Rightarrow V_z(0) = V_z$

Then $V_1(t) = V_1 \cos \omega t + k \sin \omega t$ $\Rightarrow V_3(t) = \frac{1}{\omega} \dot{V}_1 = k \cos \omega t - V_1 \sin \omega t$

$\Rightarrow \begin{align*}
\vec{r}(t) &= \frac{V_1}{\omega} \sin \omega t, \quad y(t) = \frac{V_1}{\omega} (\cos \omega t - 1) + y_0, \quad z(t) = V_1 t - A \text{ helix} \\
\text{we've chosen } \vec{r}(0) &= (0, y_0, 0). \quad \text{nb. If } y_0 = \frac{V_1}{\omega}, \text{ then particle travels in spiral (helix)}
\end{align*}$
Can recover general solution by rotating & translating coordinates.

**Alternate Solution:**

\[
\vec{V} = \omega \vec{v} \times \vec{n}, \quad \vec{n} \text{ is unit vector in direction of } \vec{B}.
\]

Note: \((\vec{\nabla} \times \vec{n})_i = E_{ijk} \vec{V}_j n_k = (E_{ijk} n_k) \vec{V}_j = (A \vec{V}_j);
\]

\[
A_{ij} = E_{ijk} n_k
\]

**nb** \(A_{ik} A_{kj} = A_{ij}^2 = E_{ikm} n_m E_{kmj} n_j = \left( \delta_{im} \delta_{mj} - \delta_{ij} \delta_{mn} \right) n_k n_m = n_i n_j - \delta_{ij} n^2 \)

\[
\vec{B} = (0, 0, B) \Rightarrow \vec{n} = (0, 0, 1)
\]

Then \(A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) & \(A^2 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

4. \(A^{2m} = (-1)^m I \quad A^{2m+1} = (-1)^m A \)

5. \(\vec{V} = \omega A \vec{V} \Rightarrow \vec{V} = e^{\omega t A} \vec{V}_0 \)

6. \(\vec{V} = \vec{V}_0 + \sum_{n=0}^{\infty} \frac{(\omega t)^{2n+1}}{(2n+1)!} \vec{V}_0 + \sum_{n=1}^{\infty} \frac{(\omega t)^{2n}}{(2n)!} (-1)^n I \vec{V}_0 \)

\[= \vec{V}_0 + \sin \omega t \vec{A} \vec{V}_0 + (\cos \omega t - 1) I \vec{V}_0 \]

7. \((x, y, z) = \left( \frac{x_0}{\omega} \sin \omega t - \frac{y_0}{\omega} \cos \omega t, \frac{y_0}{\omega} \sin \omega t + \frac{x_0}{\omega} \cos \omega t, \frac{z_0}{\omega} t \right) + (x_0, y_0, z_0) \)

**nb** \(\frac{d^2}{dt^2} = \frac{d^2}{\omega^2} (t) = \text{const.} \)
\[ \mathbf{B} = (0, 0, B) = \text{const} \quad \mathbf{E} = 0 \]

\[ \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} = \frac{B}{2} (-y, x, 0) \]

\[ \mathbf{r}^2 = (x, y, z) = (B \cos \phi, B \sin \phi, z) \]

\[ \mathbf{r}' = \mathbf{r} = \left( B \cos \phi - B \dot{\phi} \sin \phi, B \sin \phi + B \dot{\phi} \cos \phi, \dot{z} \right) \]

\[ \dot{r} = \frac{1}{2} m v^2 - \frac{e \phi}{c} + \frac{1}{2} \mathbf{v} \cdot \mathbf{A} \]

\[ \mathbf{v} \cdot \dot{\mathbf{r}} = \dot{s}^2 + \dot{\phi}^2 + \dot{z}^2 \quad \mathbf{v} \cdot \mathbf{A} = \frac{B}{2} \dot{\phi} \]

\[ L = \frac{1}{2} m \left( \dot{s}^2 + \dot{\phi}^2 + \dot{z}^2 \right) + \frac{elB}{2c} \dot{\phi}^2 \]

\[ E = \text{const} = \frac{1}{2} m v^2 = \frac{1}{2} m \left( \dot{s}^2 + \dot{\phi}^2 + \dot{z}^2 \right) \]

in cylindrical coords:

\[ \frac{dx}{dt} = \frac{d\phi}{dt} = \dot{z} = \text{const} \Rightarrow \frac{dx}{d\phi} = m \dot{z} = \text{const} \equiv m v_\perp \]

\[ L = L_z = P_\phi = m s^2 (\dot{\phi} + \frac{1}{2} \omega) = \text{const}, \quad \omega = \frac{2B}{mc} \]

Now since \( \dot{z} = v_\perp = \text{const} \), the energy factorises

\[ E = E_\perp + E_\parallel = \frac{1}{2} m (\dot{s}^2 + \dot{\phi}^2) + \frac{1}{2} m \dot{z}^2 \quad E_\perp = \text{const}, \quad E_\parallel = \text{const} \]

Note that \( \dot{z} = \text{const} \) and if we show \( \dot{s}^2 = \text{const} \) then

\[ L_z = \text{const} \quad \Rightarrow \quad \dot{\phi} = \text{const} \quad (\& \text{ vice versa}) \]

Then we are done with this solution, but, PTO.
\[ \dot{q} = \frac{L^2}{mg^2} - \frac{1}{2} \omega \]

\[ E_\perp = \frac{1}{2} M \left( \dot{q}^2 + \mathbf{S}^2 \left( \frac{L^2}{mg^2} - \frac{1}{2} \omega \right) \right) \]  
(\text{write } L = L_2)

\[ = \frac{1}{2} MS^2 + \frac{L^2}{mg^2} - \frac{Lw}{m} + \omega^2 S^2 \]

This can be solved by quadrature

\[ t - t_0 = \int_{s_0}^{s} \frac{ds}{\frac{dS}{dt}} dt \]

\text{This will yield a sinusoidal solution.}

Eventually we would find that we get the motion, \rightarrow

i.e. our origin is in the wrong place!

Examine

\[ E_\perp (S_\perp) = \frac{1}{2} M S^2 \left( \frac{L}{mg^2} - \frac{1}{2} \omega \right) \]

\[ \Rightarrow S^2 = \frac{4}{m\omega^2} \left( E_\perp + \frac{Lw}{2} \pm \sqrt{E_\perp (E_\perp + Lw)} \right) \]

Now choose origin so that \( \mathbf{r}_0 \cdot \mathbf{v}_0 = 0 \), \( z_0 = 0 \) & \( \theta_0 = 0 \)

\[ \Rightarrow 0 = (S_\perp, 0, 0) \cdot (\dot{S}_\perp, S_\perp \dot{\theta}_0, 0) = S_\perp \dot{\theta}_0 \]

If \( S_\perp \neq 0 \) then \( \dot{\theta}_0 = 0 \)

Thus

\[ E_\perp = \frac{1}{2} MS^2 \dot{\theta}_0^2 \]  \&  \[ L_\theta = MS^2 (\dot{\theta}_0 + \frac{1}{2} \omega) \]

\[ S^2 = \frac{2S^2}{\omega^2} \left( \dot{\theta}_0^2 + \omega \dot{\theta}_0 + \frac{1}{2} \omega^2 \pm \sqrt{2 \dot{\theta}_0^2 (\dot{\theta}_0 + \omega)^2} \right) \]

So if we choose \( S_\perp \) s.t. \( \dot{\theta}_0 = -\omega \) then \( \dot{S}_\perp = S_\perp \Rightarrow S = \text{const.} \)

\[ \text{nb.} \quad S = \frac{2E_\perp}{m\omega^2} \Rightarrow \frac{S}{\sqrt{E_\perp}} = \frac{V_\perp}{\omega}, \quad \text{c.f. Cartesian solution} \]

If \( \dot{S}_\perp = \text{const.} \) then \( \dot{z} = \text{const.} \) \( \dot{\theta} = \text{const.} \) & \( \dot{\phi} = -\omega = \text{const.} \)
Note on the choice of origin for cylindrical coords:

\[ \mathbf{\mathbf{r}} = (x, y, z) = (\rho \cos \varphi + a, \rho \sin \varphi + b, z + c) \]

\[ \Rightarrow \mathbf{\ddot{r}} = \mathbf{\ddot{r}} = (\rho \cos \varphi - \dot{\rho} \sin \varphi, \dot{\rho} \sin \varphi + \rho \dot{\varphi} \cos \varphi, \ddot{z}) \]

\[ \mathbf{\nabla} \cdot \mathbf{\ddot{r}} = \frac{1}{2} \rho (\dot{\rho}^2 + \dot{\varphi}^2 + \ddot{z}^2) \text{ as per normal.} \]

If \( \mathbf{B} = (0, 0, B) \Rightarrow \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} = \frac{B}{2} (-y, x, 0) \)

then \[ \mathbf{\nabla} \cdot \mathbf{A} = \frac{B}{2} (x \dot{y} - y \dot{x}) \]

\[ = \frac{B}{2} \left( \dot{\rho}^2 + \frac{\partial}{\partial \varphi} \left( a (\rho \sin \varphi + b) - b (\rho \cos \varphi + c) \right) \right) \]

\[ \mathcal{L} = \frac{1}{2} \rho (\dot{\rho}^2 + \dot{\varphi}^2 + \ddot{z}^2) + \frac{g \rho}{2} \rho \dot{\varphi}^2 + \frac{B}{2} \frac{\partial}{\partial \varphi} \left( a (\rho \sin \varphi + b) - b (\rho \cos \varphi + c) \right) \]

choice of origin does not affect the equations of motion, 

but is important to the structure of the solution.

(similarly can examine x-y relations)
Equations of motion: 
\[ m \ddot{\mathbf{r}} = q \mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} \]

\[ (m \frac{d^2}{dt^2} + \frac{q}{c} \mathbf{B} \times ) \mathbf{v} = q \mathbf{E} \]

The LHS is a homogeneous DE for \( \mathbf{v} \), the Electric field applies an inhomogeneous term.

\[ \mathbf{v} = \mathbf{v}_{\text{homog}} + \mathbf{v}_{\text{pi}} \]

where \( \mathbf{v}_{\text{homog}} \) is the general solution to the constant magnetic field case & \( \mathbf{v}_{\text{pi}} \) is any particular integral for the full equation.

Choose \( \mathbf{B} = B \hat{z} \), \( \mathbf{E} = E_\parallel \hat{z} + E_\perp \hat{y} \) \((\hat{z}, \hat{y} \text{ are unit vectors})\)

then DE \[ \ddot{x} = \omega y, \quad \ddot{y} = -\omega x + \frac{q}{m} E_\perp, \quad \ddot{z} = \frac{q}{m} E_\parallel = \text{const.} \]

\[ z = \frac{qE_\parallel}{2m} t^2 + z_0 t + z_0, \quad t_0 = 0, \quad z_0 = z(0) \]

\[ \frac{d}{dt} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \frac{qE_\parallel}{m} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \frac{d}{dt}(x + iy) = -i \omega (x + iy) + i \frac{qE_\parallel}{m} \]

A particular integral is \( (x + iy) = \frac{1}{\omega m} E_\perp = \frac{cE_\perp}{B} \)

\[ (x + iy) = e^{-i\omega t} (x_0 + iy_0) + \frac{cE_\perp}{B} t \]

\[ (x + iy) = (x_0 + iy_0) + \frac{q}{\omega} (x_0 + iy_0) e^{-i\omega t} + \frac{cE_\perp}{B} t \]

\[ t_0 = 0 \]
We can rotate the axes so that $\dot{y}_0 = \ddot{y}(0) = 0$

Then shift $t = \tau$ get $\vec{r}(0) = 0$

\[
\begin{align*}
\chi &= \frac{x}{w} \sin wt + \frac{\beta E_x}{wm} t \\
\eta &= \frac{\beta}{\omega} (\cos wt - 1) \\
\zeta &= \frac{\beta E_x}{2m} t^2 + \dot{z}_0 t
\end{align*}
\]

Note that the time average (over $\tau = \frac{2\pi}{\omega}$) of the $x$-velocity is

\[
\overline{\chi} = \frac{\beta E_x}{wm} = \frac{\beta E_x}{B} = \text{const}
\]

So $E_x \ll B$ if the non-relativistic approx. is to hold.

Now write $\dot{x}_0 = -\frac{\beta E_x}{B}$

\[
\Rightarrow \chi = \frac{CE_x}{\omega B} (\omega t - \chi \sin wt), \quad \eta = \chi \frac{CE_x}{\omega B} (1 - \cos wt)
\]

& \quad \dot{\chi} = \frac{CE_x}{B} (1 - \chi \cos wt), \quad \dot{\eta} = \chi \frac{CE_x}{B} \sin wt
\]

& \quad \frac{dy}{d\chi} = \frac{\dot{y}}{\dot{\chi}} = -\frac{\chi \sin wt}{1 - \chi \cos wt}
\]

Now examine the $x$-$y$ projection:

If $|\chi| < 1$

If $|\chi| = 1$

If $|\chi| > 1$

These are just different projections of the helix, depending on the $\vec{r}$ velocity.